CIV 2552 – Mét. Num. Prob. de Fluxo e Transporte em Meios Porosos
1º Semestre – 2011

Trab4: Método dos Elementos Finitos
Fluxo hidráulico em regime permanente 1D

Resolva a 2ª questão do segundo trabalho para regime permanente pelo Método dos Elementos Finitos utilizando:

(a) elementos finitos lineares
(b) elementos finitos quadráticos.

Complete o programa em MATLAB onde indicado. Para o modelo com elementos quadráticos, a função "preprocessor" também deverá ser modificada.

Trab4.m

```
% Include global variables
include_gblrefs;

% Preprocessing
preprocessor;

% Assembly global permeability matrix Kg
% COMPLETE AQUI

% Assembly global forcing vector F
% COMPLETE AQUI

% Solve for u
% u = Kg \ F;

% Plot steady-state solution
postprocessor;
```

include_gblrefs.m

```
% File to include global variables

% Global size parameters
global nnp                      % number of nodal points
global nel                      % number of elements
global nen                      % number of element nodes

% Nodal coordinates
global x                        % array of nodal x coordinates

% Element properties arrays
global CArea                    % array of element cross-sectional area values
global Permeability             % array of element permeability values

% Element nodal connectivity array, location matrix and gather matrix
global LM                       % gather matrix: LM(nen,nel)
                                %  stores global nodal number for
                                %  each local node of each element
```

```
% Global equation matrix, forcing vector, and solution vector
global Kg                        % global permeability matrix
global F                         % global system forcing vector
global u                         % global system solution vector

% Boundary Conditions (B.C.) information
%  flag = 0 --> natural B.C.
%  flag = 1 --> essencial B.C.
global bc_init_flag              % initial node B.C. flag
global bc_end_flag               % end node B.C. flag
global bc_init_val               % initial node B.C. value
global bc_end_val                % end node B.C. value

% Distributed and point source loads
global dist_load                 % array of element distributed source loads

% Analytical soluction data
global nasp                      % number of analytical solution points
global x_as                      % x coordinates of analytical solution points
global u_as                      % analytical field solution values
global q_as                      % analytical flux solution values
```

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preprocessor.m

```
% Preprocessor:
% input data for 1D example of 2nd question of Trab2 with 5 linear elements

function preprocessor

include_gblrefs;

% 1D domain parameters
L     = 80.0;             % length [m]
A     = 5.0;              % cross-section area [m2]
K     = 8.0e-6;           % permeability coefficient [m/s]
q     = 1.0e-6;           % distributed external source load [m/s]
H1    = 40.0;             % essencial B.C. at beginning (x = 0) [m]
H2    = 5.0;              % essencial B.C. at end (x = L) [m]

% Discretization parameters
nnp  =  6;                % number of nodal points
nel  =  5;                % number of elements
nen  =  2;                % number of element nodes

% Global equations coefficient matrices, RHS vector, and solution vector
Kg  = zeros(nnp,nnp);     % initialize global permeability matrix
F   = zeros(nnp,1);       % initialize global system forcing vector
u   = zeros(nnp,1);       % initialize global system solution vector

% Element properties vectors
CArea         = A*ones(nel,1);  % cross-section area
Permeability  = K*ones(nel,1);  % permeability coefficient

% B.C.'s
bc_init_flag = 1;
bc_end_flag  = 1;
bc_init_val  = H1;
bc_end_val   = H2;
```

```
% Element distributed source load vector
dist_load = ones(nel,1)*q;

% x coordinates array
x = zeros(nnp,1);
x = linspace(0.0,L,nnp);

% gather matrix = connectivity array
LM = zeros(nen,nel);
if( nen == 2 )
 LM(1,:) = (1:1:nnp-1);
 LM(2,:) = (2:1:nnp);
else
 LM(1,:) = (1:2:nnp-2);
 LM(2,:) = (2:2:nnp-1);
 LM(3,:) = (3:2:nnp);
end

% Steady-state analytical solution
nasp = 101;
x_as = zeros(nasp,1);
x_as = linspace(0.0,L,nasp);
for i=1:nasp
 u_as(i) = -0.0125*x_as(i)^2 + 0.5625*x_as(i) + 40.0;
 q_as(i) = -K * (-0.025*x_as(i) + 0.5625);
end
```

postprocessor.m

```
% Postprocessing steady state plots for given solution vector
% and computed flux

function postprocessor

include_gblrefs;

nsegs = 16;      % number of segments to divide element to get refined solution

% Dimension main field response arrays
% In case of quadratic elements compute refined field solution along element
% In case of linear elements, just use mesh x coordinate and field response
if( nen == 3 )
 x_field = zeros((nel*nsegs)+1,1);
 v_field = zeros((nel*nsegs)+1,1);
else
 x_field = x;
 v_field = u;
end

% Compute refined solution field along element
if( nen == 3 )
 i = 1;
 for e=1:nel
  LMe            = LM(:,e);            % extract element nodal connectivity
  xe             = x(LMe);             % extract element x coordinates
  ue             = u(LMe) ;            % extract element node main field values
  le             = xe(nen)-xe(1);      % length of element
```

```
  Dx             = le/nsegs;         % increment along element
  xs             = xe(1);            % holds point position to compute response
  for j=1:nsegs
   x_field(i) = xs;
   v_field(i) = dot( Nmatrix1D( xs, xe ), ue );
   xs = xs + Dx;
   i = i+1;
  end
  if( e == nel )
   xs = xe(nen);
   x_field(i) = xs;
   v_field(i) = dot( Nmatrix1D( xs, xe ), ue );
  end
 end
end

% Create figure for main field plots and get handle to it
fig_field = figure;

% Locate main field figure at left side of screen
screen_sizes = get(0,'ScreenSize');
fig_field_pos = get( fig_field, 'Position' );
fig_field_pos(1) = 0;
set( fig_field, 'Position', fig_field_pos );

% Plot main field response
plot(x_field,v_field,'Color','r');

% Setup labels
xlabel('x');
ylabel('u');
title('Trab4: steady-state field response');
hold on


% Dimension flux response arrays
x_flux = zeros(nel*2,1);
v_flux = zeros(nel*2,1);

% Compute flux response from given solution vector and plot it
for e=1:nel
 K              = Permeability(e);  % get element permeability coefficient
 LMe            = LM(:,e);          % extract element nodal connectivity
 xe             = x(LMe);           % extract element x coordinates
 ue             = u(LMe);           % extract element node main field values
 x_flux(2*e-1) = xe(1);             % first flux point in element is located
                                    %  at first element node
 x_flux(2*e)   = xe(nen);           % second flux point in element is located
                                    %  at last element node
 v_flux(2*e-1) = -K * dot( Bmatrix1D( x_flux(2*e-1), xe ), ue );
 v_flux(2*e)   = -K * dot( Bmatrix1D( x_flux(2*e),   xe ), ue );
end

% Create figure for flux response plots and get handle to it
fig_flux = figure;

% Locate flux results figure at right side of screen
screen_sizes = get(0,'ScreenSize');
fig_flux_pos = get( fig_flux, 'Position' );
```

```
fig_flux_pos(1) = screen_sizes(3) - fig_flux_pos(3);
set( fig_flux, 'Position', fig_flux_pos );

% Plot given solution vector
plot(x_flux,v_flux,'Color','r');

% Setup labels
xlabel('x');
ylabel('q');
title('Trab4: steady-state flux response');
hold on

% Plot analytical solutions (if available)
if( nasp )
 figure( fig_field );
 plot(x_as,u_as,'Color','k');
 figure( fig_flux );
 plot(x_as,q_as,'Color','k');
end
```

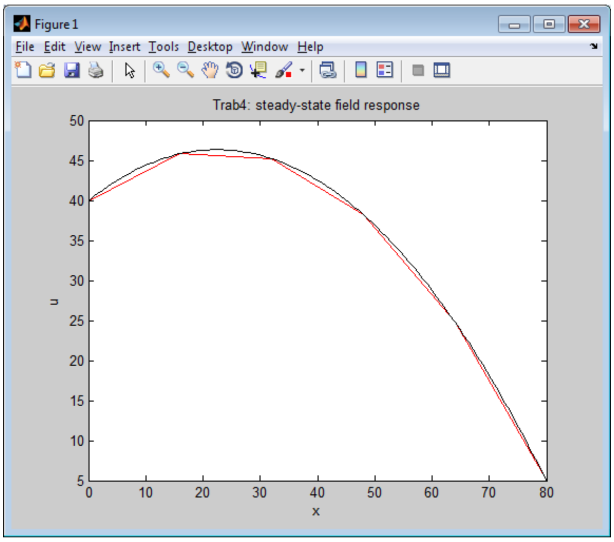
Nmatrix1D.m

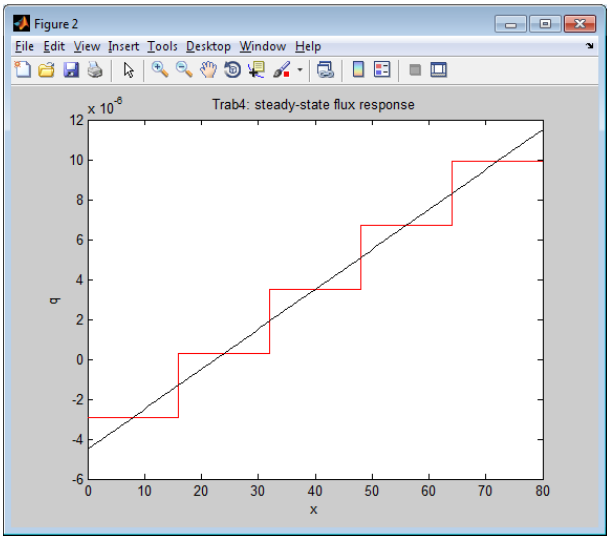
```
% Evaluates shape functions (in physical coordinates) at point xt

function N = Nmatrix1D(xt,xe)

include_gblrefs;

if nen == 2         % linear shape functions
 N(1) = (xt-xe(2))/(xe(1)-xe(2));
 N(2) = (xt-xe(1))/(xe(2)-xe(1));
elseif nen == 3     % quadratic shape functions
 N(1) = (xt-xe(2))*(xt-xe(3))/((xe(1)-xe(2))*(xe(1)-xe(3)));
 N(2) = (xt-xe(1))*(xt-xe(3))/((xe(2)-xe(1))*(xe(2)-xe(3)));
 N(3) = (xt-xe(1))*(xt-xe(2))/((xe(3)-xe(1))*(xe(3)-xe(2)));
end
```

Bmatrix1D.m

```
% Evaluates derivative of the shape functions (in physical coordinates)
% at point xt

function B = Bmatrix1D(xt,xe)

include_gblrefs;

if nen == 2       % derivative of linear shape functions
 B =  1/(xe(1)-xe(2))*[1 -1];
elseif nen == 3   % derivative of quadratic shape functions
 B(1) = (2*xt-xe(2)-xe(3))/((xe(1)-xe(2))*(xe(1)-xe(3)));
 B(2) = (2*xt-xe(1)-xe(3))/((xe(2)-xe(1))*(xe(2)-xe(3)));
 B(3) = (2*xt-xe(1)-xe(2))/((xe(3)-xe(1))*(xe(3)-xe(2)));
end
```

Imagens dos resultados (regime permanente) obtidos para o exemplo da 2ª questão do segundo trabalho utilizando 5 elementos finitos lineares:





Consideração das condições de contorno essenciais (de Dirichlet)

Uma maneira conveniente para considerar as condições de contorno essenciais do problema proposto ($h_1 = H_1$ e $h_n = H_n$) é modificar a matriz global de permeabilidade $[Kg]$ e o vetor das cargas equivalentes nodais $\{F\}$ depois de eles terem sido criados sem considerar nenhuma condição de contorno. Na formulação do problema 1D com o elemento finito linear com dois nós, a matriz $[Kg]$ tem um formato tridiagonal:

$$\begin{bmatrix} Kg_{1,1} & Kg_{1,2} & 0 & 0 & ... & 0 & 0 \\ Kg_{2,1} & Kg_{2,2} & Kg_{2,3} & 0 & ... & 0 & 0 \\ & & ... & & & & \\ & & & ... & & & \\ 0 & 0 & ... & 0 & Kg_{n-1,n-2} & Kg_{n-1,n-1} & Kg_{n-1,n} \\ 0 & 0 & ... & 0 & 0 & Kg_{n,n-1} & Kg_{n,n} \end{bmatrix} \cdot \begin{Bmatrix} H_1 \\ h_2 \\ ... \\ ... \\ h_{n-1} \\ H_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ ... \\ ... \\ F_{n-1} \\ F_n \end{Bmatrix}$$

Apenas as duas primeiras linhas e as duas últimas linhas de $[Kg]$ e de $\{F\}$ precisam são modificadas, tal como indicado abaixo:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & Kg_{2,2} & Kg_{2,3} & 0 & ... & 0 & 0 \\ & & ... & & & & \\ & & & ... & & & \\ 0 & 0 & ... & 0 & Kg_{n-1,n-2} & Kg_{n-1,n-1} & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} h_1 \\ h_2 \\ ... \\ ... \\ h_{n-1} \\ h_n \end{Bmatrix} = \begin{Bmatrix} H_1 \\ F_2 - Kg_{2,1} \cdot H_1 \\ ... \\ ... \\ F_{n-1} - Kg_{n-1,n} \cdot H_n \\ H_n \end{Bmatrix}$$

A primeira e a última linha da matriz ficam com um "1" na diagonal principal e "0" nos outros termos. Os termos de carga no vetor $\{F\}$ na primeira e na última linha têm o valor das condições de contorno essenciais H_1 e H_n, respectivamente. Para manter a simetria da matriz global, o primeiro termo da segunda linha e o último termo da penúltima linha da matriz são anulados, sendo que os termos de carga correspondentes são alterados tal como indicado, levando-se em conta que os termos anulados da matriz são os que multiplicam os valores conhecidos das condições de contorno essenciais. Dessa forma, o número de equações do sistema não se altera em relação ao número total de nós, h_1 e h_n continuam sendo incógnitas, e a solução da primeira e última linhas do sistema resulta nas condições de contorno essenciais.